

A note on the double-critical graph conjecture

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Abstract

A connected n -chromatic graph G is double-critical if for all the edges xy of G , the graph $G - x - y$ is $(n - 2)$ -chromatic. In 1966, Erdős and Lovász conjectured that the only double-critical n -chromatic graph is K_n . This conjecture remains unresolved for $n \geq 6$. In this short note, we verify this conjecture for claw-free graphs G of chromatic number 6.

1 Introduction

In this note, we consider finite and simple graphs. For a graph G , we use $V(G)$ and $E(G)$ to denote the set of vertices and edges of G , respectively. A *subgraph* of G is a graph whose vertex set is a subset of $V(G)$ and whose edge set is a subset of $E(G)$. We say that a subgraph H is an *induced subgraph* of G if, for any $x, y \in V(H)$, $xy \in E(H)$ iff $xy \in E(G)$. Let G be a graph and $S \subset V(G)$. Then $G[S]$, the subgraph of G induced by S , denotes the graph with vertex set S and edge set $\{uv \in E(G) : u, v \in S\}$, and let $G - S = G[(V(G) \setminus S)]$. When $S = \{x, y\}$, we often write $G - x - y$ instead of $G - S$. For a positive integer k , a *proper k -coloring* of a graph G is a function c from $V(G)$ to a set of k colors such that $c(u) \neq c(v)$ for any $uv \in E(G)$. A graph G is *k -colorable* if G has a proper k -coloring. We use $\chi(G)$ to denote the smallest integer k such that G is k -colorable, which is known as the *chromatic number* of G . Further we denote by $\omega(G)$ and $\alpha(G)$ the size of the largest clique and independent set in G , respectively, and $N(v)$ the set of vertices adjacent to the vertex v in G .

We say that a connected graph G with chromatic number n is *n -double-critical*, if, for any $xy \in E(G)$, $\chi(G - x - y) = n - 2$. It is easy to see that the complete graph K_n is n -double-critical. The following elegant conjecture was posed by Erdős and Lovász [3] more than fifty years ago.

Conjecture 1.1 *K_n is the only n -double-critical graph.*

It is easy to see that Conjecture 1.1 holds for $n \leq 3$. With some extra work it can also be verified for $n = 4$. In 1986, Stiebitz [7] showed that Conjecture 1.1 is true for $n = 5$. He proved the existence of K_4 in every 5-double-critical graph by considering uniquely 3-colorable subgraphs of G . However, this technique does not seem to generalize to finding a larger clique in an n -double-critical graph with $n \geq 6$.

The double-critical graph conjecture is a special case of a more general conjecture, the so-called Erdős-Lovász Tihany conjecture [3]: for any graph G with $\chi(G) > \omega(G)$ and any two integers

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$k, l \geq 2$ with $k + l = \chi(G) + 1$, there exists a partition (S, T) of the vertex set such that $\chi(G[S]) \geq k$ and $\chi(G[T]) \geq l$. The Erdős-Lovász Tihany conjecture was proved for various cases: $(k, l) = (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (3, 5)$ (see: [2, 6, 7, 8]). Kostochka and Stiebitz [4] showed that it is true for all (k, l) for line graphs. Balogh et al. [1] generalized this result to quasi-line graphs (a graph is a quasi-line graph if the neighbors of every vertex v can be expressed as the union of two cliques) and graphs of independence number 2.

A *claw* is a 4-vertex graph with one vertex of degree 3 and the others of degree 1. For convenience, we write $(v; v_1, v_2, v_3)$ to denote a claw in which v has degree 3. A graph G is *claw-free* if it does not have a claw as an induced subgraph. Note that the graphs from both families in Balogh et al.'s result [1] are claw-free. It would be interesting to know whether the double-critical graph conjecture, or the Erdős-Lovász Tihany conjecture holds for all the claw-free graphs. As a step in that direction, we prove the following theorem.

Theorem 1.2 *Let G be a double-critical graph with $\chi(G) = 6$. If G is claw-free, then $G \cong K_6$.*

The rest of this note is organized as follows. In the next section we will prove several lemmas that will be repeatedly used throughout the proof of Theorem 1.2. Section 3 contains the proof of Theorem 1.2.

2 Lemmas

Given a graph G of chromatic number n , and a proper n -coloring of G , all vertices of the same color form a *color class*. By definition, each color class is an independent set in G .

Lemma 2.1 *Let G be an n -double-critical graph. If $\omega(G) \geq n - 1$, then $G \cong K_n$. Thus if $G \not\cong K_n$, then $\omega(G) \leq n - 2$.*

Proof. Suppose $\omega(G) \geq n - 1$. Let $v_1, \dots, v_{n-1} \in V(G)$ be the vertices that induce a copy of K_{n-1} . Among all the proper n -colorings of G with color classes V_1, \dots, V_n , and $v_i \in V_i$ for $1 \leq i \leq n - 1$, we choose one that minimizes $|V_n|$.

Let $v_n \in V_n$. We claim that $N(v_n) \cap V_i \neq \emptyset$ for $i = 1, \dots, n - 1$. Suppose not. Without loss of generality we may assume that $N(v_n) \cap V_1 = \emptyset$. If $V_n = \{v_n\}$ then the independent sets $V_1 \cup \{v_n\}, V_2, \dots, V_{n-1}$ form a proper $(n - 1)$ -coloring of G , contradicting the assumption that $\chi(G) = n$. If $V_n \setminus \{v_n\} \neq \emptyset$, then $V_1 \cup \{v_n\}, V_2, \dots, V_n \setminus \{v_n\}$ are the color classes of an n -coloring of G . Thus we have a contradiction to the minimality of $|V_n|$.

We now show that v_n is adjacent to every vertex in $\{v_1, \dots, v_{n-1}\}$. Suppose not. Without loss of generality assume that $v_1 \notin N(v_n)$. Then, by the above claim, v_n is adjacent to some $y \in V_1 \setminus \{v_1\}$. However $\chi(G - v_n - y) = n - 2$, a contradiction, since $\{v_1, \dots, v_{n-1}\}$ induces a copy of K_{n-1} in $G - v_n - y$.

Hence $\{v_1, \dots, v_n\}$ induces K_n in G . If $V(G) = \{v_1, \dots, v_n\}$, then $G \cong K_n$. Suppose $V(G) \neq \{v_1, \dots, v_n\}$, and let $x \in V(G)$ such that $x \notin \{v_1, \dots, v_n\}$. Since G is connected, there exists $z \in V(G)$ such that $xz \in E(G)$. However, $G - x - z$ contains a clique on $n - 1$ vertices, which contradicts that $\chi(G - x - z) = n - 2$, and thus $G \cong K_n$. ■

Lemma 2.2 *Let G be an n -double-critical graph that is claw-free. For $xy \in E(G)$, let V_1, \dots, V_{n-2} be the color classes of an $(n-2)$ -coloring of $G - x - y$. Then $N(x) \cap N(y) \cap V_i \neq \emptyset$ for $i \in \{1, \dots, n-2\}$.*

Proof. Suppose not. Then, without loss of generality, we may assume that $N(x) \cap N(y) \cap V_{n-2} = \emptyset$. Let $V_x = \{x\} \cup (V_{n-2} \setminus N(x))$ and $V_y = \{y\} \cup V_{n-2} \setminus V_x$. Note that V_x and V_y are independent sets. Now $V_1, \dots, V_{n-3}, V_x, V_y$ are the color classes of an $(n-1)$ -coloring of G , contradicting the assumption that $\chi(G) = n$. ■

The degree of a vertex v , denoted $d(v)$, in a graph is the number of edges incident to v . We denote by $\Delta(G)$ and $\delta(G)$ the maximum and minimum degree of a vertex in G respectively. The following lemma shows that for $\chi(G) = 6$, it suffices to consider 6-double-critical graphs in which every pair of adjacent vertices has 4 or 5 common neighbors.

Lemma 2.3 *Let G be a 6-double-critical graph that is also claw-free. If $G \not\cong K_6$, then for any $xy \in E(G)$, $4 \leq |N(x) \cap N(y)| \leq 5$. If, in addition, $|N(x) \cap N(y)| = 5$, then $G[N(x) \cap N(y)] \cong C_5$.*

Proof. We may assume that $\Delta(G[N(x) \cap N(y)]) \leq 2$. For, otherwise, let $v \in N(x) \cap N(y)$ and let $v_1, v_2, v_3 \in N(x) \cap N(y)$ such that $vv_i \in E(G)$, $i = 1, 2, 3$. Since $(v; v_1, v_2, v_3)$ does not induce a claw in G , there exist $i, j \in \{1, 2, 3\}$ such that $i \neq j$ and $v_i v_j \in E(G)$. Thus $\{v, v_i, v_j, x, y\}$ induces a copy of K_5 in G . Hence by Lemma 2.1, $G \cong K_6$.

We may also assume that $\delta(G[N(x) \cap N(y)]) \geq |N(x) \cap N(y)| - 3$. For, otherwise, let $v \in N(x) \cap N(y)$ and let $v_1, v_2, v_3 \in N(x) \cap N(y)$ such that $vv_i \notin E(G)$, $i = 1, 2, 3$. Then $v_i v_j \in E(G)$ for all distinct $i, j \in \{1, 2, 3\}$, since $(v; v_i, v_j, x)$ does not induce a claw in G . Therefore $\{v_1, v_2, v_3, x, y\}$ induces a copy of K_5 in G . Once again by Lemma 2.1, $G \cong K_6$.

Hence, if $G \not\cong K_6$, $|N(x) \cap N(y)| - 3 \leq \delta(G[N(x) \cap N(y)]) \leq \Delta(G[N(x) \cap N(y)]) \leq 2$. Thus $|N(x) \cap N(y)| \leq 5$. On the other hand, by Lemma 2.2, $|N(x) \cap N(y)| \geq 6 - 2 = 4$.

Now suppose $|N(x) \cap N(y)| = 5$. Then $2 = |N(x) \cap N(y)| - 3 \leq \delta(G[N(x) \cap N(y)]) \leq \Delta(G[N(x) \cap N(y)]) \leq 2$. Hence $\delta(G[N(x) \cap N(y)]) = \Delta(G[N(x) \cap N(y)]) = 2$, so every vertex in $G[N(x) \cap N(y)]$ has degree 2. The only 2-regular graph on 5 vertices is C_5 , thus $G[N(x) \cap N(y)] \cong C_5$. ■

The following lemma is an easy consequence of G being claw-free.

Lemma 2.4 *Let G be a claw-free graph, and S an independent set of G . Suppose $x \in V(G) \setminus S$. Then $|N(x) \cap S| \leq 2$.*

Proof. Suppose $|N(x) \cap S| \geq 3$. Let $x_1, x_2, x_3 \in N(x) \cap S$. Since S is an independent set, $(x; x_1, x_2, x_3)$ induces a claw in G , a contradiction. ■

3 Proof of the Main Result

Theorem 1.2 follows from the two lemmas in this section. From Lemma 2.3, we may assume that the number of common neighbors of any two adjacent vertices is either 4 or 5. The first lemma settles the case when there exists a pair with 4 common neighbors.

Lemma 3.1 *Let G be a 6-double-critical graph that is claw-free. If $|N(x) \cap N(y)| = 4$ for some $xy \in E(G)$, then $G \cong K_6$.*

Proof. For an arbitrary $xy \in E(G)$, by Lemma 2.3, we have $|N(x) \cap N(y)| \geq 4$. Thus $d(x) \geq 5$ and $d(y) \geq 5$. Moreover, if V_1, V_2, V_3, V_4 denote the color classes of a 4-coloring of $G - x - y$, it follows from Lemma 2.4 that $|N(x) \cap V_i| \leq 2$ and $|N(y) \cap V_i| \leq 2$ for $i \in \{1, 2, 3, 4\}$. Thus $d(x) \leq 9$ and $d(y) \leq 9$.

Claim 1. If $xy \in E(G)$ and $|N(x) \cap N(y)| = 4$, then $d(x), d(y) \in \{7, 8\}$.

Let $N(x) \cap N(y) = \{v_1, v_2, v_3, v_4\}$, and let V_1, V_2, V_3, V_4 be the color classes of a 4-coloring of $G - x - y$. By Lemma 2.2, we may assume $v_i \in V_i$, $i = 1, 2, 3, 4$.

Suppose $d(x) \in \{5, 6\}$. Then we may assume that $N(x) \cap V_i = v_i$ for $i \in \{2, 3, 4\}$. Since $|N(x) \cap N(v_1)| \geq 4$, $v_1 v_i \in E(G)$ for $i \in \{2, 3, 4\}$. If, for every pair of distinct $i, j \in \{2, 3, 4\}$, $v_i v_j \notin E(G)$, then $(v_1; v_2, v_3, v_4)$ induces a claw in G . Otherwise, there exist distinct $i, j \in \{2, 3, 4\}$ such that $v_i v_j \in E(G)$. Then $G[\{v_1, v_i, v_j, x, y\}] \cong K_5$. Hence $G \cong K_6$ by Lemma 2.1.

Now suppose that $d(x) = 9$. Let $N(x) \setminus \{v_1, v_2, v_3, v_4, y\} = \{u_1, u_2, u_3, u_4\}$. By Lemma 2.4, we may assume $u_i \in V_i$ for $i \in \{1, 2, 3, 4\}$. For any distinct $j, k \in \{1, 2, 3, 4\}$, $u_j u_k \in E(G)$; for otherwise $(x; u_j, u_k, y)$ induces a claw. Thus $G[\{x, u_1, u_2, u_3, u_4\}] \cong K_5$. Hence, by Lemma 2.1, $G \cong K_6$.

Claim 2. If $xy \in E(G)$ and $|N(x) \cap N(y)| = 4$, then $d(x) = d(y) = 8$.

Let $N(x) \cap N(y) = \{v_1, v_2, v_3, v_4\}$, and let V_1, V_2, V_3, V_4 be the color classes of a 4-coloring of $G - x - y$. By Lemma 2.2, we may assume $v_i \in V_i$, $i = 1, 2, 3, 4$.

Suppose $d(x) = 7$ and, by Lemma 2.4 and by symmetry, let $u_i \in N(x) \cap V_i \setminus v_i$, $i \in \{1, 2\}$. Since $|N(x) \cap N(u_1)| \geq 4$ and $u_1 \notin N(y)$, $u_1 u_2, u_1 v_2, u_1 v_3, u_1 v_4 \in E(G)$. Similarly, since $|N(x) \cap N(u_2)| \geq 4$ and $u_2 \notin N(y)$, $u_2 v_1, u_2 v_3, u_2 v_4 \in E(G)$.

We claim that y does not have a neighbor in $V_1 \setminus \{v_1\}$ or $V_2 \setminus \{v_2\}$. For otherwise, suppose there exists $w_1 \in V_1 \setminus v_1$ such that $y w_1 \in E(G)$. Since $|N(y) \cap N(w_1)| \geq 4$, we have $|N(w_1) \cap \{v_2, v_3, v_4\}| \geq 2$. (This is because $d(y) \leq 8$, so y has at most two neighbors not from $\{x, v_1, v_2, v_3, v_4, w_1\}$. If w_1 is adjacent to at most one vertex from $\{v_2, v_3, v_4\}$, then $|N(y) \cap N(w_1)| \leq 3$.) Similarly since $|N(x) \cap N(v_1)| \geq 4$, we have $|N(v_1) \cap \{v_2, v_3, v_4\}| \geq 2$. Thus there exists $i \in \{2, 3, 4\}$ such that $v_i \in N(v_1) \cap N(w_1)$.

Note $(v_i; v_1, u_1, w_1)$ induces a claw in G , a contradiction. Therefore by Claim 1 and Lemma 2.4, $d(y) = 7$ and $|N(y) \cap V_i| = 2$ for $i \in \{3, 4\}$. Let $w_i \in N(y) \cap V_i \setminus \{v_i\}$ for $i \in \{3, 4\}$. Since $|N(y) \cap N(w_3)| \geq 4$ and $w_3 \notin N(x)$, $w_3 w_4, w_3 v_1, w_3 v_2, w_3 v_4 \in E(G)$, and similarly since $|N(y) \cap N(w_4)| \geq 4$ and $w_4 \notin N(x)$, $w_4 v_1, w_4 v_2, w_4 v_3 \in E(G)$.

We may assume that $v_3 v_4 \notin E(G)$; otherwise $G[\{x, u_1, u_2, v_3, v_4\}] \cong K_5$ and, hence, $G \cong K_6$ by Lemma 2.1. Similarly we may assume $v_1 v_2 \notin E(G)$, otherwise $G[\{y, v_1, v_2, w_3, w_4\}] \cong K_5$ and once again $G \cong K_6$ by Lemma 2.1. Since $|N(x) \cap N(v_1)| \geq 4$ and $v_1 v_2 \notin E(G)$, $v_1 v_3, v_1 v_4 \in E(G)$. Similarly since $|N(x) \cap N(v_2)| \geq 4$ and $v_1 v_2 \notin E(G)$, $v_2 v_3, v_2 v_4 \in E(G)$.

We claim that $u_i w_j \in E(G)$ for all $i \in \{1, 2\}$ and $j \in \{3, 4\}$. Suppose not. By symmetry, we assume $w_4 u_1 \notin E(G)$. Since $|N(v_2) \cap N(w_4)| \geq 4$, from the known adjacencies we so far only have $w_3, v_3, y \in N(v_2) \cap N(w_4)$, therefore there exists $w_1 \in V_1 \cup V_3$ such that $w_1 v_2, w_1 w_4 \in E(G)$. In fact, $w_1 \in V_1$, otherwise then $(w_4; v_3, w_1, w_3)$ induces a claw in G , a contradiction. Since $w_4 u_1 \notin E(G)$, $w_1 \neq u_1$. Note that $w_1 v_3 \notin E(G)$ otherwise $(v_3; v_1, u_1, w_1)$ induces a claw and similarly $w_1 v_4 \notin E(G)$ otherwise $(v_4; v_1, u_1, w_1)$ induces a claw. Then $(v_2; w_1, v_3, v_4)$ induces a claw in G , a contradiction since G is claw-free. Hence, $G[\{u_1, u_2, w_3, w_4\}] \cong K_4$.

Consider the graph $G - x - v_3$. Since G is 6-double-critical, $\chi(G - x - v_3) = 4$. Let $c : V(G) \rightarrow \{1, 2, 3, 4\}$ be a 4-coloring of $G - x - v_3$. Since $G[\{u_1, u_2, w_3, w_4\}] \cong K_4$, we may assume $c(u_1) = 1$,

$c(u_2) = 2$, $c(w_3) = 3$, $c(u_4) = 4$. Then $c(v_1) = 1$, since $v_1u_2, v_1w_3, v_1w_4 \in E(G)$. Similarly $c(v_2) = 2$ and $c(v_4) = 4$. Then, since $yv_1, yv_2, yw_3, yw_4 \in E(G)$, y cannot be colored by any of the colors 1, 2, 3, 4. Thus $G - x - v_3$ is not 4-colorable, a contradiction. Therefore $d(x) = 8$. Similarly $d(y) = 8$. This completes the proof of Claim 2.

Now let us fix $xy \in E(G)$ with $|N(x) \cap N(y)| = 4$. Let $N(x) \cap N(y) = \{v_1, v_2, v_3, v_4\}$ and let V_1, V_2, V_3, V_4 be the color classes of a 4-coloring of $G - x - y$. By Lemma 2.2, we may assume $v_i \in V_i$ for $i \in \{1, 2, 3, 4\}$. By Claim 2, $d(x) = 8$. Thus let $w_i \in N(x) \cap V_i \setminus \{v_i\}$, $i \in \{1, 2, 3\}$. Note that $G[\{w_1, w_3, w_4\}]$ is a clique of size 3; otherwise suppose, by symmetry $w_1w_2 \notin E(G)$, then $(x; y, w_1, w_2)$ induces a claw in G .

Claim 3. For $i \in \{1, 2, 3\}$, $\{v_1, v_2, v_3, v_4\} \setminus \{v_i\} \subset N(w_i)$.

Suppose otherwise. Without loss of generality, we may assume $\{v_2, v_3, v_4\} \not\subset N(w_1)$. Since $|N(x) \cap N(w_1)| \geq 4$, $|N(w_1) \cap \{v_2, v_3, v_4\}| = 2$, and $|N(x) \cap N(w_1)| = 4$. By Claim 2, it suffices to consider the case $d(w_1) = 8$. However, from Lemma 2.4,

$$d(w_1) \leq 1 + |N(w_1) \cap V_2| + |N(w_1) \cap V_3| + |N(w_1) \cap V_4| \leq 7,$$

a contradiction. Hence we have Claim 3.

Note that for $i \in \{1, 2, 3\}$, $v_iv_4 \notin E(G)$ otherwise $\{x, v_i, v_4, w_1, w_2\}$ induces a K_5 . To avoid the claw $(y; v_i, v_j, v_4)$, we must have $v_iv_j \in E(G)$ for distinct $i, j \in \{1, 2, 3\}$. In this case $\{x, y, v_1, v_2, v_3\}$ induces a copy of K_5 , and hence $G \cong K_6$ and the proof of Lemma 3.1 is complete. \blacksquare

Our next lemma settles the remaining case when every pair of adjacent vertices have exactly 5 common neighbors.

Lemma 3.2 *Let G be 6-double-critical graph, and assume that G is claw-free. Suppose $|N(x) \cap N(y)| \geq 5$ for all $xy \in E(G)$. Then $G \cong K_6$.*

Proof. We prove this Lemma by way of contradiction. Suppose $G \not\cong K_6$. Then, by Lemma 2.1, $K_5 \not\subset G$. By Lemma 2.3 and Lemma 3.1, we may assume that $|N(x) \cap N(y)| = 5$ for all $xy \in E(G)$. Let $N(x) \cap N(y) = \{v_1, v_2, v_3, v_4, v_5\}$, and let V_1, V_2, V_3, V_4 be the color classes of a 4-coloring of $G - x - y$. By Lemma 2.2, we may assume that $v_i \in V_i$, $i \in \{1, 2, 3, 4\}$ and $v_5 \in V_1$. By Lemma 2.3, $\{v_1, v_2, v_3, v_4, v_5\}$ induces a C_5 in G . Without loss of generality, assume $v_1v_2, v_1v_4, v_2v_5, v_3v_4, v_3v_5 \in E(G)$ and $v_1v_3, v_2v_3, v_2v_4, v_4v_5 \notin E(G)$.

Since $|N(x) \cap N(v_5)| = 5$, there exist $a, b \in (N(x) \cap N(v_5)) \setminus \{v_1, v_2, v_3, v_4, v_5, y\}$. Similarly since $|N(y) \cap N(v_5)| = 5$, there exist $c, d \in (N(y) \cap N(v_5)) \setminus \{v_1, v_2, v_3, v_4, v_5, x\}$. Note that $a, b, c, d \in (V_2 \cup V_3 \cup V_4) \setminus \{v_2, v_3, v_4\}$. Since $N(x) \cap N(y) = \{v_1, v_2, v_3, v_4, v_5\}$, $a, b \notin N(y)$ and $c, d \notin N(x)$. Hence a, b, c, d are pairwise distinct. Moreover $ab \in E(G)$ to avoid the claw $(x; a, b, y)$ in G , and $cd \in E(G)$ to avoid the claw $(y; c, d, x) \in E(G)$. By Lemma 2.4 (applied to v_5 and V_i , for $i \in \{2, 3, 4\}$) and by the symmetry between V_2 and V_3 , we may assume that $b, d \in V_4$, $a \in V_3$, and $c \in V_2$.

Since $|N(y) \cap N(v_3)| \geq 5$, there exist $z_1, z_2 \in (N(y) \cap N(v_3)) \setminus \{v_1, v_2, v_3, v_4, v_5, x, y\}$. Note that $z_i \notin V_1$ for $i \in \{1, 2\}$; otherwise $(y; v_1, v_5, z_i)$ induces a claw in G . Clearly $z_1, z_2 \notin V_3$ since V_3 is independent.

We claim that $d \in \{z_1, z_2\}$. Suppose otherwise, $d \notin \{z_1, z_2\}$. In this case $z_i \notin V_4$ for $i \in \{1, 2\}$ to avoid the claw $(y; d, v_4, z_i)$ in G . Therefore $z_1, z_2 \in V_2$, then $(y; v_2, z_1, z_2)$ induces a claw in G , a contradiction.

Since $d \in \{z_1, z_2\}$, we have $dv_3 \in E(G)$. By a similar argument considering $|N(x) \cap N(v_3)| \geq 5$, $bv_3 \in E(G)$. Thus $(v_3; b, d, v_4)$ induces a claw in G , a contradiction. ■

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